If $w \neq 0$ equation (1) shows that the velocities of all points P relative to any one point O are at right angles to OP and proportional to the distance OP. For this reason the scalar w is called the ANGULAR VELOCITY of the plane motion.

4. If $w \neq 0$ we can find a point of the figure whose velocity is instantaneously zero. For if this point be called C we have from (1) the condition $0-\dot{O}=wi(C-O)$; dividing by w and multiplying by i we find:

$$C = O + \frac{1}{w} i\dot{O}. \tag{2}$$

There exists only one such point. For, dividing (1) by w and multiplying it by i we have

$$\frac{1}{w} i\dot{P} - \frac{1}{w} i\dot{O} = -P + O,$$

whence

$$P + \frac{1}{w} i\dot{P} = O + \frac{1}{w} i\dot{O} = C,$$

by (2); i. e., the same point C is reached from whichever point P of the figure we may start.

This point C of zero velocity is called the INSTANTANEOUS CENTER of the plane motion.

5. If the velocity $\dot{O}$ of any point O and the angular velocity w are given, the instantaneous center C is found, according to (2), by turning $\dot{O}$ about O through $\pi/2$ and multiplying it by $1/w$.

It follows that C lies on the normal of the path of every point of the figure and hence can also be found as the intersection of the normals to the paths of any two points whose velocities are not parallel (Fig. 2).

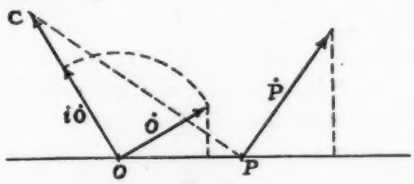


FIG. 2.

If in (1) we take for the arbitrary point O the instantaneous center C whose velocity is zero we find:

$$\dot{P} = wi(P - C); \tag{3}$$

i. e., the velocity of every point P of the figure is perpendicular to its radius vector drawn from C and proportional to its distance from C; in other words, *if not a translation* (all velocities equal, $w = 0$), *the instantaneous motion of a plane rigid figure in its plane is a rotation about the instantaneous center, with angular velocity* w.

6. If the velocities $\dot{O}, \dot{P}$ of any two points O, P of the figure are given (subject to the condition of Art. 1), the velocity $\dot{Q}$ of any other point Q of the figure can be found without constructing the instantaneous center. For, the fundamental relation (1) shows that if the vectors $\dot{O}, \dot{P}$ be drawn from the same origin, say O' (Fig. 3), the line-segment $O'P'$ joining their extremities is perpendicular to OP and w times OP. It follows that the extremities of the velocities of any number of points of the figure, if drawn from the same origin, will form a figure similar to that of the points (though turned through a right angle), the ratio of similarity being w.

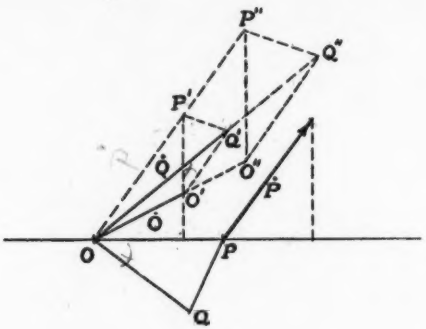


FIG. 3.

Hence, to find $\dot{Q}$ from $\dot{O}$ and $\dot{P}$ make (Fig. 3) $O' - O = \dot{O}$, $P' - O = \dot{P}$, and construct on $O'P'$ a triangle $O'P'Q'$ similar to OPQ, by drawing $O'Q'$ at right angles to OQ and $P'Q'$ at right angles to PQ; then $Q' - O = \dot{Q}$.

If we had used for the velocities a unit $= 1/w$ times the unit of length, making (Fig. 3) $O'' - O = (1/w)\dot{O}$, $P'' - O = (1/w)\dot{P}$, we should have $O''P'' = OP$ and triangle $O''P''Q''$ congruent to triangle OPQ; O is the center of similarity of the triangles $O'P'Q'$ and $O''P''Q''$.

7. Another method for deriving $\dot{Q}$ from $\dot{O}$ and $\dot{P}$ consists in using projections, according to the last statement of Art. 1. The projections of the required velocity $\dot{Q}$ on OQ and PQ must be equal, respectively, to the projections of $\dot{O}$ on OQ and

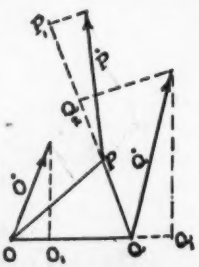


FIG. 4.

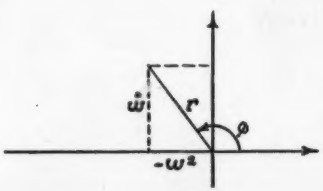


FIG. 5.

of P on PQ; and from these known projections Q can be found (Fig. 4): make $QQ_1 = OO_1$, $QQ_2 = PP_1$, and draw through Q_1 a perpendicular to OQ, through Q_2 a perpendicular to PQ.

II. Accelerations.

8. The fundamental relation (1) gives upon differentiation with respect to the time:

$$\ddot{\boldsymbol{p}} = wi\dot{\boldsymbol{p}} + \dot{w}i\boldsymbol{p} = -w^2\boldsymbol{p} + \dot{w}i\boldsymbol{p}, \tag{4}$$

i. e.,

$$\ddot{P} = \ddot{O} + (-w^2 + \dot{w}i)(P - O).$$

Let us write the complex number $-w^2 + \dot{w}i$ in the polar form (Fig. 5):

$$-w^2 + \dot{w}i = re^{\varphi i},$$

so that

$$r = \sqrt{w^4 + \dot{w}^2}, \qquad \tan\varphi = \frac{\dot{w}}{-w^2}.$$

Taking r positive we have $\frac{1}{2}\pi \leqq \varphi \leqq \frac{3}{2}\pi$ and $\varphi \gtreqless \pi$ according as $\dot{w} \gtreqless 0$. The case of translation when both w and $\dot{w}$ are zero is here excluded.

We then have simply

$$\ddot{\boldsymbol{p}} = re^{\varphi i}\boldsymbol{p}, \quad \text{or} \quad \ddot{P} = \ddot{O} + re^{\varphi i}(P - O); \tag{4'}$$

i. e., *the acceleration $\ddot{P}$ of any point P is found from the acceleration $\ddot{O}$ of any other point O by adding to $\ddot{O}$ the vector obtained by turning $P - O$ through the angle φ and multiplying it by r,* φ and r being, at a given instant, the same for all points P of the figure.

9. As $r \neq 0$ (the case of translation being excluded) we can always find one and only one point, say A, whose acceleration is zero. For, the condition $0 = \ddot{O} + re^{\varphi i}(A - O)$ gives

$$A = O - \frac{1}{r}e^{-\varphi i}\ddot{O}. \tag{5}$$

This point A of zero acceleration is called the CENTER OF ACCELERATION.

If in the general relation (4') we select for O the center of acceleration we find

$$\ddot{P} = re^{\varphi i}(P - A); \tag{6}$$

i. e., *the acceleration $\ddot{P}$ of any point P is obtained by turning the radius vector of P, drawn from A, through the angle φ and multiplying it by r.*

The acceleration $\ddot{P}$ has therefore two components: one, w^2 mod $(P - A)$, directed toward A, the other, $\dot{w}$ mod $(P - A)$, at right angles to AP.

All points of a circle about A have accelerations of equal magnitude and equally inclined to their radii vectores drawn from A; all points of a line through A have parallel accelerations, in magnitude proportional to the distance from A.

10. If, in particular, $\dot{w} = 0$ we have (Art. 8) $r = w^2$, $\varphi = \pi$, so that (5) and (6) reduce to

$$A = O + \frac{1}{w^2}\ddot{O}, \qquad \ddot{P} = -w^2(P - A).$$

Thus, *if the angular velocity w is constant (but $\neq 0$), the accelerations of all points are directed toward one and the same point, the center of acceleration, and are proportional to the distance from this point.*

If, on the other hand, $w = 0$ while $\dot{w} \neq 0$, we have if $\dot{w}$ is positive: $r = \dot{w}$, $\varphi = \frac{1}{2}\pi$, and if $\dot{w}$ is negative: $r = -\dot{w}$, $\varphi = \frac{3}{2}\pi$, so that (5) and (6) reduce to

$$A = O + \frac{1}{\dot{w}} i\ddot{O}, \qquad \ddot{P} = \dot{w}i(P - A).$$

Thus, *if the angular velocity w is zero (while $\dot{w} \neq 0$), the accelerations of all points are at right angles to their radii vectores drawn from the center of acceleration and proportional to the distance from it.*

11. Comparing the equations (1) and (4′) it appears that the field of acceleration vectors, like that of velocity vectors, is completely determined by any two vectors. But in the case of the velocities the vectors $\dot{O}$, $\dot{P}$ cannot be selected arbitrarily since (Art. 1) $\dot{\boldsymbol{p}} \times \boldsymbol{p} = 0$, i. e., $\dot{O}$, $\dot{P}$ must have equal projections on OP. In the case of the accelerations the vectors $\ddot{O}$, $\ddot{P}$ can be taken arbitrarily.

Multiplying (4) by $\times \boldsymbol{p}$ we find

$$\ddot{\boldsymbol{p}} \times \boldsymbol{p} = -w^2\boldsymbol{p}^2; \tag{7}$$

i. e., *the projections of the accelerations $\ddot{O}$, $\ddot{P}$ of any two points O, P of the figure on the line OP differ by a quantity (w^2 mod $\boldsymbol{p}$) proportional to the distance OP of the points.*

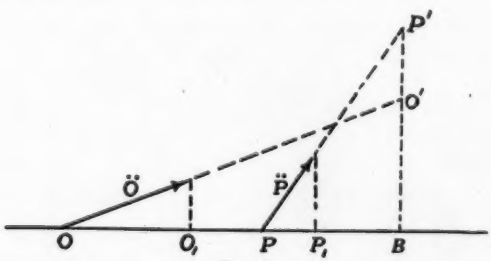


FIG. 6.

To exhibit this relation graphically let us take the unit of acceleration as $1/w^2$ times the unit of length; the difference of the projections of the acceleration vectors, drawn on this scale, will then be equal to the distance OP; i. e., if these vectors be drawn from O and P, the projections of their extremities on OP will coincide, say at B.

Thus, if OO_1, PP_1 (Fig. 6) are the projections on OP of the actual accelerations

(drawn on the same scale as the distances), and if $O' - O = (1/w^2)\ddot{O}$, $P' - P = (1/w^2)\ddot{P}$, the projections of O' and P' on OP coincide at B. As OB is divided by O_1 in the same ratio (viz., w^2) in which PB is divided by P_1, we have

$$\frac{OB}{OO_1} = \frac{OB - OP}{PP_1}, \qquad \text{whence} \qquad OB = \frac{OO_1}{OO_1 - PP_1} \cdot OP,$$

or putting $PP_1/OO_1 = m$:

$$OB = \frac{1}{1 - m} OP.$$

12. This leads to a simple *construction of the center of acceleration A*. Indeed, A is the intersection (different from B) of the circles described about OO' and PP' as diameters (Fig. 7).

For, any point Q on the circle about OO' has its acceleration at right angles to OQ since the projection of OO' on OQ is OQ; and similarly any point R of the circle about PP' has its acceleration at right angles to PR. Hence the acceleration of A, which would have to be perpendicular to both OA and PA, must be zero.

13. The acceleration of O, if represented by $O' - O$, has the rectangular

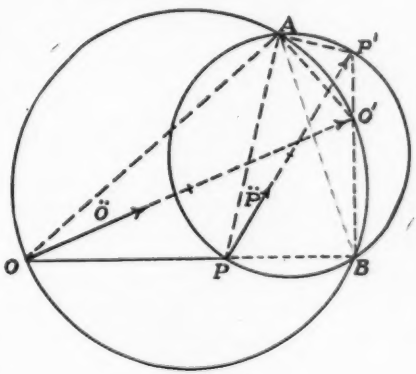


FIG. 7.

components $A - O$ and $O' - A$; similarly the acceleration $P' - P$ of P has the rectangular components $A - P$ and $P' - A$. Now (Fig. 7) $\measuredangle OO'A = OBA$, $\measuredangle OBA = PBA = PP'A$; hence $\measuredangle OO'A = PP'A$. The rightangled triangles OAO', PAP' are therefore similar, and hence

$$\frac{OA}{AO'} = \frac{PA}{AP'}.$$

This means that if the acceleration of any point be resolved along and at right

angles to the radius vector drawn from the center of acceleration, these components have the same ratio for all points of the figure. This, of course, also follows from formula (6) since $\measuredangle AOO'$ is the supplement of φ.

The above construction of the center of acceleration follows also from formula (5) which can be written:

$$A = O + \frac{1}{r} e^{(\pi-\phi)i} \ddot{O} = O + \frac{w^2}{r} e^{(\pi-\phi)i} \cdot \frac{1}{w^2} \ddot{O},$$

or since $w^2/r = -\cos\varphi$(Art. 8):

$$A = O - \cos\varphi \cdot e^{(\pi-\phi)i} \cdot \frac{1}{w^2} \ddot{O}.$$

14. It follows from the preceding articles that *if the accelerations $\ddot{O}$, $\ddot{P}$ of any two points O, P of the figure are given, the acceleration $\ddot{Q}$ of any other point Q can be found,* provided $w \neq 0$. For, by Art. 11 we can find the point B, and from this by Art. 12 the center of acceleration A; hence, constructing on AQ a triangle AQQ' similar to AOO' we find

$$\frac{1}{w^2} \ddot{Q} = Q' - Q, \quad \text{where} \quad w^2 = \frac{\operatorname{mod} \ddot{O}}{\operatorname{mod} (O' - O)}.$$

This construction cannot be used when $w = 0$. But in this case (Art. 10) A is found as the intersection of perpendiculars through O, P to $\ddot{O}$, $\ddot{P}$, respectively. If these perpendiculars happen to be parallel, A lies at infinity; if they coincide, A is found as the intersection of OP with the line joining the extremities of $\ddot{O}$ and $\ddot{P}$.

15. The acceleration $\ddot{Q}$ of any point Q of the figure can however be found from the accelerations $\ddot{O}$, $\ddot{P}$ of any two points O, P without first constructing the center of acceleration. The construction follows directly from the relations $\ddot{\boldsymbol{p}} = re^{\phi i}\boldsymbol{p}$, $\ddot{\boldsymbol{q}} = re^{\phi i}\boldsymbol{q}$, where $\boldsymbol{q} = Q - O$ (Fig. 8). The vectors $P - O = \boldsymbol{p}$ and $\ddot{P} - \ddot{O} = \ddot{\boldsymbol{p}}$,

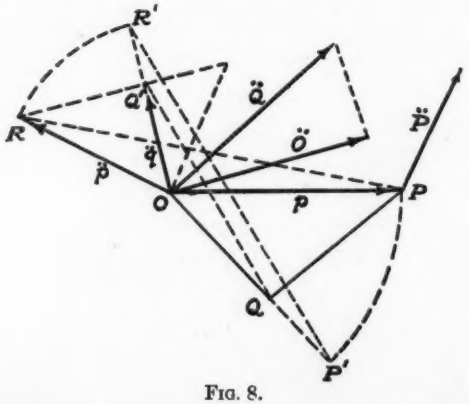


FIG. 8.

drawn from O, form a triangle OPR with the angle φ at O. Turn this triangle about O through an angle $= \measuredangle POQ$ so that it takes the position $OP'R'$; a parallel through Q to $P'R'$ then determines $\ddot{q} = Q' - O$. Adding this to $\ddot{O}$ we find $\ddot{Q}$.

16. Finally, by using projections somewhat as in Art. 7 for velocities, we can derive $\ddot{Q}$ from $\ddot{O}$ and $\ddot{P}$ as follows.

Let OO_1 and OP_1 be the projections on OP of $\ddot{O}$ and $\ddot{P}$, drawn from O (Fig. 9), and construct on P_1O_1 the triangle $P_1O_1Q_1$ similar to OPQ. Let the parallel to OP through Q_1 meet OQ at Q', PQ at Q''; and let OO' be the projection of $\ddot{O}$ on OQ, PP' the projection of $\ddot{P}$ on PQ.

Then $Q'O'$ is the projection of $\ddot{O}$ on QO, $Q''P'$ the projection of $\ddot{Q}$ on QP; transferring these to Q, $\ddot{Q}$ is found.

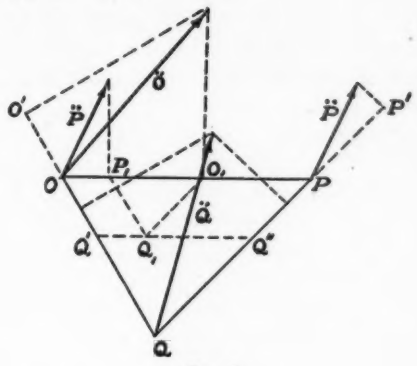


FIG. 9.

The constructions of Arts. 7, 11–13, 16 were first obtained from the equations in Cartesian coordinates, OP being taken as axis of x. Using the vector method it is naturally preferable to deal with the vectors themselves rather than with their projections.

NOTE ON CERTAIN ALGEBRAIC EQUATIONS.

By HERMON L. SLOBIN, University of Minnesota.

In elementary analysis we often meet problems like the following:
Show that the roots of the equation

I. $$x^3 + x^2 - 2x - 1 = 0 \quad \text{are} \quad 2\cos\frac{2}{7}\pi, \quad 2\cos\frac{4}{7}\pi, \quad 2\cos\frac{8}{7}\pi.$$

Show that the roots of the equation

II. $$x^3 - 5x^2 + 6x - 1 = 0 \quad \text{are} \quad 4\cos^2\frac{\pi}{7}, \quad 4\cos^2\frac{2\pi}{7}, \quad 4\cos^2\frac{4\pi}{7}.$$

These problems are taken from Bromwich, "Theory of Infinite Series," page 188. The method of solution used by Bromwich involves a knowledge of certain formulæ in the theory of trigonometric series. These formulæ are based upon the consideration of a certain differential equation, and the method of solution by their means is applicable only when these roots of the algebraic equations have very special forms.

Problems may be stated which require algebraic equations to be set up whose roots are given trigonometric forms such as the above. In a previous paper[1] it has been shown that the trigonometric functions whose arguments are rational multiples of π, are algebraic numbers. The general problem would then be to find the algebraic equation whose roots are $\theta_1(r_1\pi)$, $\theta_2(r_2\pi)$, $\cdots$ $\theta_n(r_n\pi)$, where θ_i represents any trigonometric function, $i = 1, 2, \cdots n$, and r_i represents any rational number. More generally, θ_i may represent any expression built up of the trigonometric functions whose arguments are rational multiples of π, by means of the operations of addition, subtraction, multiplication, division, raising to powers, and extracting roots.

A formal elementary method is readily obtainable for the solution of such problems, but in the general case the operations of reducing the resultant equation to an algebraic equation with rational coefficients would be quite cumbersome, and the degree of the final equation often much larger than the number of roots assigned as such trigonometric forms. However, in particular cases, where the degree of the rational algebraic equation is the same as the number of roots assigned as trigonometric forms, the operations are simple, and the *process* is applicable even in the general case.

I shall illustrate this method by applying it to the problems stated above. The detailed operations are instructive and the forms in which the coefficients enter in the equation are of especial interest.

1. Given the roots $a_1 = 2\cos\frac{2\pi}{7}$, $a_2 = 2\cos\frac{4\pi}{7}$, $a_3 = 2\cos\frac{8\pi}{7}$.

From De Moivre's theorem, $(\cos x + i\sin x)^r = \cos rx + i\sin rx$, when $x = \pi$, and r is any rational number, we have

$$\cos r\pi = \frac{(-1)^{2r} + 1}{2(-1)^r},$$

where $(-1)^r$ denotes *any of the complex roots*. That is, if $r = p/q$, $(-1)^{p/q}$ means the pth power of any of the complex qth roots of -1. Hence

$$a_1 = \frac{(-1)^{\frac{1}{7}} + 1}{(-1)^{\frac{1}{7}}}, \quad a_2 = \frac{(-1)^{\frac{2}{7}} + 1}{(-1)^{\frac{1}{7}}}, \quad a_3 = \frac{(-1)^{\frac{4}{7}} + 1}{(-1)^{\frac{1}{7}}}.$$

The equation

$$x^3 - (a_1 + a_2 + a_3)x^2 + (a_1a_2 + a_1a_3 + a_2a_3)x - a_1a_2a_3 = 0$$

will now be constructed.

[1] Bauer and Slobin, *Rendiconti*, 1913.

Multiplying numerator and denominator of a_1 by $(-1)^{\frac{1}{7}}$ and reducing the numerator by substituting (-1) for $(-1)^{\frac{1}{7}}$, we have

$$a_1 = (-1)^{\frac{2}{7}} - (-1)^{\frac{1}{7}};$$

similarly

$$a_2 = (-1)^{\frac{4}{7}} - (-1)^{\frac{2}{7}};$$
$$a_3 = -(-1)^{\frac{4}{7}} + (-1)^{\frac{3}{7}}.$$

Hence

$$a_1 + a_2 + a_3 = [(-1)^{\frac{2}{7}} - (-1)^{\frac{3}{7}} + (-1)^{\frac{4}{7}} - (-1)^{\frac{5}{7}} + (-1)^{\frac{6}{7}} - (-1)^{\frac{1}{7}}].$$

We will denote this bracket by K. From $x^7 + 1 = 0$ we have for all the complex roots of $(-1)^{\frac{1}{7}}$,

$$x^6 - x^5 + x^4 - x^3 + x^2 - x = -1.$$

Hence

$$K = -1.$$

Likewise

$$a_1a_2 = (-1)^{\frac{3}{7}} + (-1)^{\frac{4}{7}} - (-1)^{\frac{5}{7}} - (-1)^{\frac{3}{7}},$$
$$a_1a_3 = -(-1)^{\frac{5}{7}} + (-1)^{\frac{4}{7}} - (-1)^{\frac{3}{7}} + (-1)^{\frac{2}{7}},$$
$$a_2a_3 = -(-1)^{\frac{6}{7}} + (-1)^{\frac{5}{7}} - (-1)^{\frac{4}{7}} + (-1)^{\frac{3}{7}}.$$

Hence

$$a_1a_2 + a_1a_3 + a_2a_3 = 2K = -2,$$

and finally

$$a_1a_2a_3 = 2 + K = 1.$$

Therefore the required equation is

$$x^3 + x^2 - 2x - 1 = 0.$$

2. Given the roots $a_1 = 4\cos^2\frac{\pi}{7}$, $a_2 = 4\cos^2\frac{2\pi}{7}$, $a_3 = 4\cos^2\frac{4\pi}{7}$. By a process similar to the above, we have

$$a_1 = (-1)^{\frac{2}{7}} - (-1)^{\frac{3}{7}} + 2,$$
$$a_2 = (-1)^{\frac{4}{7}} - (-1)^{\frac{2}{7}} + 2,$$
$$a_3 = -(-1)^{\frac{4}{7}} + (-1)^{\frac{5}{7}} + 2.$$

Also,

$$a_1 + a_2 + a_3 = 6 + K = 5,$$
$$a_1a_2 + a_1a_3 + a_2a_3 = 12 + 6K = 6,$$
$$a_1a_2a_3 = 10 + 9K = 1.$$

Hence the equation is

$$x^3 - 5x^2 + 6x - 1 = 0.$$

As a further example a case involving somewhat more difficult rationalizations is proposed by the author in the problem department in this issue, namely, to find the algebraic equation whose roots are

$$a_1 = \cos\frac{\pi}{9},\ a_2 = -\cos\frac{2\pi}{9},\ a_3 = -\cos\frac{4\pi}{9}.$$

A PLEA FOR LESS FORMAL WORK IN MATHEMATICS.

By F. M. MORGAN, Dartmouth College.

An American professor of mathematics while abroad was told by one of his English colleagues, that American mathematical text-books looked as if they had been written for the feeble-minded. The American professor replied: "Perhaps that is so but your text-books look as if they had been written by the feeble-minded." Now I feel like Sir Roger de Coverly when he said "Much may be said on both sides." However in this brief paper I intend to discuss the question only from the foreign professor's point of view.

First I ask why should our American texts as a whole call for such a remark? I believe we do not have to look far in order to find the answer. It lies in the fact that the majority of them contain too may rules and formulæ. They are written chiefly with the idea of cramming a student to pass an examination and not with the idea of teaching him mathematics. The stimulus for original thinking is therefore sacrificed, everything being done by a cut and dried rule. Many of our texts make mathematics a subject for memory instead of a subject for reason. They try to feed the student mathematics in the form of little sugar-coated pills.

There are of course certain rules and formulæ which are fundamental to the science. These must be taught to, and memorized by, the student. There are others which are not fundamental and should not be taught, for generally these non-important rules and formulæ hold as prominent a place in the student's mind as the important ones do. Let me illustrate my point by a concrete example. A large number of our American text-books on algebra give the following rule for evaluating a third order determinant: "Rewrite the first and second columns to the right of the determinant. The diagonals running down from left to right give the positive terms. The diagonals running down from right to left give the negative terms." Now this rule is so simple that any child can use it, but I ask, of what real good is the rule? It does not apply to a determinant of any other order, and therefore it is not a rule fundamental to the study of determinants. Expansion by means of minors is just as easy, and it is, moreover, fundamental to the subject since it may be applied to a determinant of any order.

Then we have rules in which the emphasis on the fundamental principle is sacrificed in order to make the rule short so that a student without mental effort may solve a problem of that type. Let me also illustrate this point by means of the usual rule for finding the equation of a straight line through a given point and parallel to a given line: "*Step* 1. Change the constant term in the given equation to k. *Step* 2. Substitute the coördinates of the given point in the equation of step 1 and solve for k. *Step* 3. Substitute this value of k in the equation of step 1, and the result is the desired equation." This rule, like the one on determinants, is easy to learn and apply, but what real knowledge of analytic geometry has the student gained by learning it? How many students

really know why the above rule gives the desired result and not, for example, the equation of a line through the given point and perpendicular to the given line? Very few, it will be found, really know. The reason for this is that the fundamental principle involved, namely that parallel lines have the same slope but different intercepts on the y axis, has been mentioned implicitly and not explicitly. If such rules as the above (and there are many more of the same type) were written so that the fundamental principle involved were explicitly stated, the student would benefit greatly thereby.

Experience makes it certain that a student learns more mathematics, can pass a better examination (if that is a criterion for excellency in the subject) if he knows the fundamental formulæ, the fundamental principles, and has been taught to reason and not memorize.

BOOK REVIEWS.

EDITED BY W. H. BUSSEY.

Advanced Algebra. By JOSEPH V. COLLINS. American Book Company, New York. x + 342 pages. $1.00.

The subject matter for a freshman course in algebra as presented in the great majority of text books does not vary much. This book differs slightly from the usual algebra in this respect. It is divided into three parts. Part I, containing five chapters, is devoted to a review of the elementary algebra through simultaneous quadratics. At the end of chapter four is a list of the fundamental principles. This is followed by a short explanation of the most common errors arising from not following these principles. The fifth chapter contains a list of the theorems of plane geometry used in the rest of the book.

Part II is made up of the chapters on graphs, ratio and proportion, logarithms, the progressions, annuities, the binomial theorem, and inequalities. The author makes evident the connection between algebra and trigonometry by following proportion with the definitions of sine, cosine, tangent and cotangent, and applying these to problems involving right triangles.

In Part III are found the topics which belong to advanced algebra proper; namely, theory of equations, permutations and combinations, probability, determinants, series, undetermined coefficients, continued fractions, and complex numbers. Some of these topics are discussed very briefly. The student is supposed to get many of the principal facts of the subject from simple, concise proofs, yet he is not burdened with too much theory of the kind which does not appeal to the average freshman. In dealing with the complex number, the addition theorem of trigonometry is derived for use in establishing De Moivre's theorem. A further connection between algebra and trigonometry is thus made.

At various places short historical notes are added. These consist, for the most part, of biographical sketches of great mathematicians, and are so placed

that they follow the subject with which the man's name is intimately associated. The chapter on logarithms ends with short notes on Briggs and Napier, that on the theory of equations with one on Gauss, determinants with one on Sylvester. A few pictures of mathematicians are also introduced.

GEO. W. HARTWELL.

Theory of Functions of a Complex Variable. BY DR. H. BURKHARDT, Professor in the Technical School, Munich. Authorized English translation, with the addition of figures and exercises, by S. E. RASOR, Professor of Mathematics, Ohio State University. D. C. Heath & Co., Boston, 1913. 421 pages. $4.00.

This book is a close translation of the second part of the first volume of Professor Burkhardt's *Funktionen-Theorie.* Since the latter is no doubt well known to those interested in this field, it will perhaps be sufficient here to give the chapter headings and to indicate briefly the additions made by the translator. Chapters, sections, and theorems are numbered exactly as in the German text.

Chapter I. *Complex numbers and their geometrical representation.* The translator has added at the end of the chapter 32 exercises designed to give practice in the use of the complex number.

Chapter II. *Rational functions and conformal representations determined by them.* Nine lists of exercises, containing 78 individual problems, have been added. Also, following § 21 there has been added § 21*a*, 7 pages, devoted to the function $w = \frac{1}{2}(z + z^{-1})$; and following § 22 there has been added § 22*a*, 6 pages, treating the function $w = z^3 - 3z$.

Chapter III. *Theory of real variables and their functions.* Four lists containing 31 exercises have been added.

Chapter IV. *Single-valued analytic functions.* A brief section, § 30*a*, on limits of convergent sequences of complex numbers, and ten lists containing 119 exercises have been added by the translator.

Chapter V. *Many-valued analytic functions.* The translator has added § 57*a* on the function $\tan^{-1} z$, § 60*a* on rational functions of z and $\sqrt{z}$, § 62*a* on rational functions of z and $\sqrt{(z-a)(z-b)}$, § 62*b* on integrals of rational functions of z, etc., § 62*c* on the function $z = w + i\sqrt{1 - w^2}$, § 62*d* on the function $\sin^{-1} w$, 27 pages in all, and six lists containing 67 exercises.

Chapter VI. *General theory of functions.* Three lists containing 48 exercises have been added.

This is without doubt a most timely book, for the need of a text in English of about this scope has long been felt. It is to be regretted that some discussion of the logarithmic potential and of streamings in general was not added for the benefit of students of physics who study the theory of functions for its applications. However, with nearly four hundred exercises added to round out in many important particulars the excellent work of Burkhardt, we now have available in English a most satisfactory text-book on the theory of functions of a complex variable.

W. C. BRENKE.

The Principles of Projective Geometry. By J. L. S. HATTON. Cambridge University Press, 1913. x + 366 pages. $3.50.

In the preface to this book the author asserts that the study of pure geometry is at present neglected and expresses the opinion that this is a misfortune. The book has been written with the hope that it may do something to alter this situation and with confidence that "the great landmarks of projective geometry" may be presented to the student in such a manner that real enthusiasm for this branch of mathematics may be aroused. The author has succeeded admirably in his attempt to write a text which gives a clear and interesting exposition of the principal methods and more important theorems of projective geometry.

The mathematical preparation with which a student is equipped when he begins the study of projective geometry is usually based on metrical concepts. Consequently the first difficulties he encounters are those arising from his being introduced all at once into an entirely new geometry from which the idea of measurement has been almost altogether excluded. The author has been aware of these difficulties and has obviated them by leading the student gradually into the non-metric geometry by the aid of metric properties. Thus the student becomes acquainted with anharmonic ratio early in the book; harmonic ratio is first defined as a special case of anharmonic ratio, and its non-metric properties and the construction of a harmonic range (pencil) from the harmonic property of a complete quadrangle (quadrilateral) are given later; the definition of involution based on its metric property is given before the non-metric definition. However, the second definition is given immediately after the first and it is proved that the metric definition might have been deduced from the second. Such a gradual transition into the study of projective geometry seems to be the best method of approach for an introductory course. The discussions and proofs are clearly presented, though enough is usually omitted to require the student to do considerable thinking for himself. The value of the book as a text is greatly enhanced by the many well-selected problems which it contains.

The first three chapters are devoted to the definitions of such fundamental concepts as range of points, pencil of lines, projection, perspective, projective relationships, anharmonic ratio and harmonic ratio. The principle of duality is here discussed and proofs of several of the properties of anharmonic and harmonic ratios, and of Ceva's and Menelaus's theorems are given. The author has used good judgment in these introductory chapters both as to the selection and the arrangement of the material.

Chapter IV contains a vivid exposition of conical (central) projection and of plane perspective. To show more clearly the differences and the similarities existing between conical projection and plane perspective, the constructions of corresponding figures and the proofs of the fundamental theorems of the two have been placed side by side.

In the next four chapters the following topics are taken up: applications of conical projection and of plane perspective to the proofs of a number of interesting theorems on triangles, quadrangles and quadrilaterals; applications of harmonic

ratio to the study of ranges and pencils in perspective, and of projective and superposed ranges and pencils; the harmonic properties of the complete quadrangle and quadrilateral; properties of involution ranges and pencils, particularly those properties of involution which are used in subsequent chapters on the conic.

In order that the matter may be more readily grasped by the student, the relations of projective forms, anharmonic properties, pole and polar, Carnot's, Pascal's and Desargues's theorems, to the circle are studied before these relations are taken up for the general conic. In the chapters on the conic some of the theorems already proved for the circle are simply restated, others are proved by methods similar to those used for circles, while in several cases alternative proofs —such as proofs by projection—are given. These chapters also contain the classification of conics and discussions of center, foci, diameters and asymptotes. The remaining eight chapters of the book deal for the most part with applications of anharmonic and harmonic ratios, involution, pole and polar, Carnot's, Pascal's and Desargues's theorems to further deductions of properties of triangles, of quadrangles and quadrilaterals, and of conics. The large number of applications and problems contained in these chapters cannot fail to interest the student of mathematics.

The addendum contains fifteen theorems and their proofs on non-projective properties of the straight line and circle. There are two very complete indexes, one of terms and definitions and the other of theorems.

The typographical errors are few and of such obvious nature as to cause the student no difficulty. The type is large and distinct and the figures have been carefully constructed.

It is the opinion of the reviewer that this book is better adapted for use as a text in a first course on projective geometry than any other book in the English language. It may also be used for reference in a lecture course on the subject. The book should do much to revive interest in the study of projective geometry in the undergraduate curriculum and "to encourage the student not to neglect the methods of pure geometry."

C. E. Stromquist.

PROBLEMS AND SOLUTIONS.

B. F. Finkel, Chairman of the Committee.

PROBLEMS FOR SOLUTION.

Special Notice.—Please reread the requests as to form of solutions on pp. 258–259 of the October 1913 issue. Unless these directions are observed by contributors, solutions must either be entirely rewritten by the committee or else rejected. Put all drawings on separate sheets.

Managing Editor.

ALGEBRA.

When this issue was made up, solutions had been received for 401, 402, 403, 404, 405, and 407. A solution of 406 is desired.

408. Proposed by EMMA M. GIBSON, Drury College.

Show that if n is a positive integer, the sum of the series

$$1 - \frac{2n-1}{1!} + \frac{(2n-1)(2n-2)}{2!} - \cdots + (-1)^{n-1}\frac{(2n-1)(2n-2)\cdots(n+1)}{(n-1)!}$$

is

$$\frac{(-1)^{n-1}(2n-2)(2n-3)\cdots(n+1)n}{(n-1)!}.$$

[From C. Smith's *Treatise on Algebra.*]

409. Proposed by C. E. GITHENS, Wheeling, W. Va.

Find integral values for the edges of a rectangular parallelopiped so that its diagonal shall be rational.

410. Proposed by C. N. SCHMALL, New York City.

Solve the simultaneous equations,

$$x^2 + xy + y^2 = a,$$

$$x^4 + x^2y^2 + y^4 = b.$$

411. Proposed by V. M. SPUNAR, Chicago, Illinois.

Determine $x_1, x_2, x_3, \cdots x_p$, from the equations

$$x_1 + x_2 + x_3 + \cdots + x_p = a_0$$

$$b_1x_1 + b_2x_2 + b_3x_3 + \cdots + b_px_p = a_1$$

$$b_1^2x_1 + b_2^2x_2 + b_3^2x_3 + \cdots + b_p^2x_p = a_2$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$b_1^{p-1}x_1 + b_2^{p-1}x_2 + b_3^{p-1}x_3 + \cdots + b_p^{p-1}x_p = a_{p-1}.$$

412. Proposed by H. L. SLOBIN, University of Minnesota.

Form the algebraic equation whose roots are: $a_1 = \cos\frac{\pi}{9}$, $a_2 = -\cos\frac{2\pi}{9}$, $a_3 = -\cos\frac{4\pi}{9}$.

(See article in this issue, page 113.)

GEOMETRY.

When this issue was made up, solutions had been received for 431, 434 and 435. Solutions of 430, 432 and 436 are desired.

Those interested in 427 may add to the footnote attached thereto in the January issue the following reference: *Das Problem der Kreisteilung*, von A. Mitzscherling, Leipzig and Berlin, 1913.

437. Proposed by J. BROOKS SMITH, Hampden-Sidney, Va.

Let D, E, F be three arbitrary points taken on the sides of a triangle ABC. If Δ and Δ' be the areas of the triangles ABC and DEF, show that

$$\frac{\Delta'}{\Delta} = \frac{AF \cdot BD \cdot CE + AE \cdot CD \cdot BF}{abc},$$

the sign of each factor being determined as follows: Each segment adjacent to one of the vertices of the triangle ABC is to be regarded as positive or negative according as it is drawn towards or from the other vertex on the side containing the segment.

438. Proposed by CLIFFORD N. MILLS, Brookings, South Dakota.

By means of the theorem that the product of the diagonals of a quadrilateral inscribed in a circle is equal to the sum of the two products of pairs of opposite sides, obtain the usual formulæ for $\sin(\alpha \pm \beta)$ and $\cos(\alpha \pm \beta)$ in terms of $\sin\alpha$, $\sin\beta$, $\cos\alpha$, $\cos\beta$ (Godfrey and Siddon's *Geometry*, page 82).

439. Proposed by CHARLES N. SCHMALL, New York City.

Show that the areas of any two triangles circumscribed about the same circle are in the same ratio as their perimeters.

440. Proposed by W. L. WATSON, Moundsville, West Va.

A solid sector is cut out of a sphere 10 feet in radius by a cone whose vertical angle is 120°. Find the radius of the sphere whose volume is equal to that of the sector.

CALCULUS.

When this issue was made up, solutions had been received for 344-5-6-7, 349, 351, 354, 356, 357, and 358. Solutions of 332, 337, 340, 342, 348, 350, 352, and 353 are desired. A complete solution of 339 is also desired.

359. Proposed by W. D. CAIRNS, Oberlin College.

Examine for maxima and minima

$$f(x) = e^{-cx}(1 + \cos x) \quad (c > 0).$$

360. Proposed by ELMER SCHUYLER, Brooklyn, New York.

What interpretation must be given to

$$\frac{d^{1/2}y}{dx^{1/2}} \text{ so that } \frac{d^{1/2}}{dx^{1/2}}\left(\frac{d^{1/2}y}{dx^{1/2}}\right) = \frac{dy}{dx}?$$

361. Proposed by EMMA M. GIBSON, Drury College.

Determine the system of curves satisfying the differential equation

$$[(1 + x^2)^{1/2} + ny]dx + [(1 + y^2)^{1/2} + nx]dy = 0,$$

and show that the curve which passes through the point $(0, n)$ contains as part of itself the conic

$$x^2 + y^2 + 2xy(1 + n^2)^{1/2} = n^2.$$

(From Forsyth's *Differential Equations*, p. 41.)

362. Proposed by C. N. SCHMALL, New York City.

Having given $y^3 - a^2y + axy - x^3 = 0$, show by Maclaurin's theorem that

$$y = -\frac{x^3}{a^2} - \frac{x^4}{a^3} - \frac{x^5}{a^4} - \cdots.$$

MECHANICS.

When this issue was made up, solutions had been received for 273, 284-5, 288, and 289. Solutions of 266, 268, 269, 271, 274-5, 277, 279, and 286 are desired.

290. Proposed by B. F. FINKEL, Drury College.

A fox, pursued by a hound, is running with uniform velocity over a frail arch in the form of a cycloid; the hound stops at a weak point of the arch, then tumbles through and reaches the level ground with a velocity equal to that of the fox. Prove that the fox exerted no normal pressure on the arch at the point where the hound fell through.

(From Walton's *Problems in Theoretical Mechanics*, p. 605.)

291. Proposed by EMMA M. GIBSON, Drury College.

The time of descent, down a rough inclined plane, of a spherical shell which contains a smooth solid sphere of the same material as itself is t_1. The time of descent, down the same plane, of a solid sphere of the same material and radius as the shell is t_2. Determine the thickness of the shell.

From Loudon's *Elementary Theory of Rigid Dynamics*, p. 188.

292. Proposed by C. N. SCHMALL, New York City.

In a bombardment, a battleship directs its fire at a fort standing on a hill whose height is a feet above the sea level. The angle of elevation of the fort is found to be ϕ. If the initial velocity of the projectile is v, show that the fort will *not* be struck if $v < \sqrt{ag(1 + \csc \phi)}$.

NUMBER THEORY.

When this issue was made up, solutions had been received for 200, 203, 206, 207 and 210. Solutions of 189, 191, 192, 196, 202, 204–5, and 208–9 are desired.

211. Proposed by E. T. BELL, Seattle, Washington.

If an odd perfect number exists, the total number of its divisors is a multiple of 2, but not of 4; or, what is the same thing, an odd perfect number must be of the form $p^{2a-1}n^2$, where p is prime and a is odd.

212. Proposed by ELMER SCHUYLER, Brooklyn, New York.

Given any positive integer N greater than 1; to prove that the sum of all the positive integers less than N and relatively prime to N equals $\frac{1}{2}N \cdot \varphi(N)$.

213. Proposed by R. D. CARMICHAEL, Indiana University.

Prove that no relatively prime integers x and y exist such that the difference of their fourth powers is a cube.

214. Proposed by A. J. KEMPNER, University of Illinois.

Let a be a positive integer $\geqq 2$, and let $T(n)$ denote the number of distinct divisors of the positive integer n, including both 1 and n, so that $T(1) = 1$, $T(2) = 2$, $T(3) = 2$, $T(4) = 3$, ... Show that

$$\Sigma_{n=1}^{n=\infty} T(n)/a^n = \Sigma_{n=1}^{n=\infty} 1/(a^n - 1).$$

The special case $a = 10$ gives, as is easily seen:

$$9 \sum_{n=1}^{n=\infty} \frac{T(n)}{10^n} = \frac{1}{1} + \frac{1}{11} + \frac{1}{111} + \frac{1}{1111} + \cdots.$$

SOLUTIONS OF PROBLEMS.

ALGEBRA.

396. Proposed by H. E. TREFETHEN, Colby College.

Show that $1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \cdots = \sqrt{2}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right)$.

I. SOLUTION BY HORACE OLSON, Chicago, Illinois.

Let S_1 represent the sum of the series, $x + \frac{x^3}{3} - \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} + \frac{x^{11}}{11} - \cdots$, of which the first member of the proposed equation is a particular case. Let S_2 represent the sum of the series $x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \frac{x^{11}}{11} + \cdots$. By calculus,

$$S_1 = \int_0^x (1 + x^2 - x^4 - x^6 + x^8 + x^{10} - \cdots)dx = \int_0^x \left(\frac{1 + x^2}{1 + x^4}\right) dx$$

$$= \frac{\tan^{-1}(\sqrt{2}x + 1) + \tan^{-1}(\sqrt{2}x - 1)}{\sqrt{2}}.$$

Also,

$$S_2 = \int_0^x (1 - x^2 + x^4 - x^6 + x^8 - x^{10} + \cdots)dx = \int_0^x \left(\frac{1}{1+x^2}\right)dx = \tan^{-1} x.$$

Letting $x = 1$, we find

$$S_1 = 1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \cdots = \frac{\tan^{-1}(\sqrt{2}+1) + \tan^{-1}(\sqrt{2}-1)}{\sqrt{2}}$$

$$= \frac{\tan^{-1}(\sqrt{2}+1) + \cot^{-1}(\sqrt{2}+1)}{\sqrt{2}} = \frac{\pi}{2\sqrt{2}}.$$

Similarly, we find for $x = 1$,

$$S_2 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots = \tan^{-1} 1 = \frac{\pi}{4}.$$

Therefore

$$\frac{\pi}{2\sqrt{2}} = 1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \cdots = \sqrt{2}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots\right).$$

II. Solution by Hermon L. Slobin, University of Minnesota.

If we develop $f(x) = 1$ as a Fourier Series

$$f(x) = a_1 \sin x + a_2 \sin 2x + \cdots + a_m \sin mx + \cdots,$$

where

$$a_m = \frac{2}{\pi}\int_0^\pi f(x) \sin mx dx,$$

we note that

$$\int_0^\pi \sin mx dx = \frac{1}{m}(1 - \cos m\pi) = 0 \quad \text{or} \quad \frac{2}{m},$$

according as m is even or odd. Hence,

$$1 = \frac{4}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right).$$

Letting $x = \pi/4$, we have

$$\frac{\pi}{4} = \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{1}{3} - \frac{\sqrt{2}}{2}\cdot\frac{1}{5} - \frac{\sqrt{2}}{2}\cdot\frac{1}{7} + \cdots\right) = \frac{\sqrt{2}}{2}\left(1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \cdots\right).$$

Therefore, the left member of the proposed identity $= \sqrt{2} \cdot \pi/4$.

But we note that, since

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots,$$

upon putting $x = 1$, we have $\pi/4 = 1 - 1/3 + 1/5 - \cdots$ which is the series in

the right member of the proposed identity. Hence

$$1+\frac{1}{3}-\frac{1}{5}-\frac{1}{7}+\frac{1}{9}+\frac{1}{11}-\cdots=\sqrt{2}\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots\right).$$

III. Solution by J. W. Campbell, University of Chicago.

Let S = sum of the series,

$$\sin 2\theta-\frac{1}{2}\sin 4\theta+\frac{1}{3}\sin 6\theta-\cdots.$$

Then S is the coefficient of i in $e^{2i\theta}-\frac{1}{2}e^{4i\theta}+\frac{1}{3}e^{6i\theta}-\cdots$, that is, in $\log_e(e^{2i\theta}+1)$, or in $\log_e e^{i\theta}(e^{i\theta}+e^{-i\theta})$, or finally in $i\theta+\log_e(2\cos\theta)$, which is θ itself.

Take $\theta=\pi/8$. Then

$$\frac{\pi}{8}=\frac{1}{\sqrt{2}}-\frac{1}{2}+\frac{1}{3\sqrt{2}}+0-\frac{1}{5\sqrt{2}}+\frac{1}{6}\cdots$$

$$=\frac{1}{\sqrt{2}}\left(1+\frac{1}{3}-\frac{1}{5}-\frac{1}{7}+\frac{1}{9}\cdots\right)-\frac{1}{2}\left(1-\frac{1}{3}+\frac{1}{5}\cdots\right).$$

That is,

$$1+\frac{1}{3}-\frac{1}{5}-\frac{1}{7}+\frac{1}{9}+\cdots=\sqrt{2}\left[\frac{\pi}{8}+\frac{1}{2}\left(1-\frac{1}{3}+\frac{1}{5}\cdots\right)\right]. \tag{1}$$

By Gregory's Series (Loney's *Trigonometry*, Part II, § 94),

$$\theta=\tan\theta-\frac{1}{3}\tan^3\theta+\frac{1}{5}\tan^5\theta-\cdots,\quad\left(-\frac{\pi}{4}\leqq\theta\leqq\frac{\pi}{4}\right).$$

Take $\theta=\pi/4$. Then $\pi/4=(1-1/3+1/5-\cdots)$. Hence,

$$\frac{\pi}{8}=\frac{1}{2}\left(1-\frac{1}{3}+\frac{1}{5}-\cdots\right). \tag{2}$$

Therefore, from (1) and (2),

$$1+\frac{1}{3}-\frac{1}{5}-\frac{1}{7}+\frac{1}{9}+\cdots=\sqrt{2}\left(1-\frac{1}{3}+\frac{1}{5}-\cdots\right).$$

Also solved by C. E. Horne, A. L. McCarty, A. M. Harding, David F. Kelley, Elmer Schuyler, C. N. Schmall, and J. W. Clawson.

ALGEBRA.

397. Proposed by W. H. BUSSEY, University of Minnesota.

12 oxen are turned into a pasture of $3\frac{1}{3}$ acres and eat all the grass in 4 weeks so that the pasture is bare. 21 oxen are turned into a pasture of 10 acres and eat all the grass in 9 weeks. How many oxen will eat all the grass of 24 acres in just exactly 18 weeks, it being assumed that the grass in all the pastures is at the same height when the oxen are turned in, and that the grass grows at a uniform rate.

I. Solution by Christian Hornung, Tiffin, Ohio.

Let x = number of pounds of grass each ox eats per week,
y = number of pounds of grass on each acre at first,
z = number of pounds of grass that grows per week per acre,
k = the number of oxen required in the last condition.

Then $\frac{10}{3}y + \frac{40}{3}z = 48x$, $10y + 90z = 189x$, and $24y + 432z = 18kx$. Eliminating y, we find $10z = 9x$, and $560z = (30k - 576)x$, from which k is found to be 36, the required number of oxen.

II. Solution by Artemas Martin, Washington, D. C.

In the first case one ox eats 1/4 of $3\frac{1}{3}/12$ or 5/72 of an acre, and 5/18 of the growth of that acre, in one week. In the second case one ox eats 1/9 of 10/21 or 10/189 of an acre, and 10/21 of what grows on an acre, in one week.

Since one ox eats the same quantity of grass in one week in each case, therefore, $10/21 - 5/18 = 25/126$ of the growth of one acre during one week is $5/72 - 10/189 = 25/1512$ of an acre; and $25/1512 \div 25/126 = 1/12$ of an acre, what grows on an acre during one week.

$5/72 + 5/18$ of $1/12 = 5/54$, the part of the original quantity of grass on one acre which one ox eats in one week.

$5/54 \times 18 = 5/3$, the quantity of grass, in acres, one ox will eat in 18 weeks.

$24 + (1/12 \times 24 \times 18) = 60$, the quantity of grass, in acres, to be eaten from 24 acres in 18 weeks; and $60 \div 5/3 = 36$, the number of oxen required to eat it.

For other solutions, see my paper on "The 'Pasturage Problem,'" published in the *Mathematical Magazine*, Vol. I, No. 2 (April, 1882), pp. 17–22; also, No. 3 of same volume, pp. 43–44.

Also solved by Albert N. Nauer, M. E. Graber, G. W. Hartwell, H. C. Feemster, Daniel Kreth, Elmer Schuyler, Horace Olson, S. W. Reaves, P. Peñalver, and J. W. Clawson.

GEOMETRY.

425. Proposed by V. M. Spunar, Chicago, Illinois.

Find the ratio of the areas A_1 and A_2 of the parabolas formed by projectiles whose ranges are the same and whose angles of projection are complements of each other.

Solution by H. C. Feemster, York College, York, Nebraska.

Let $gx^2 - 2v^2x \cos\theta_1 \sin\theta_1 + 2yv^2 \cos^2\theta_1 = 0$ and $gx^2 - 2v^2x \cos\theta_2 \sin\theta_2 + 2yv^2 \cos^2\theta_2 = 0$ be the two required parabolas, where $\theta_1 + \theta_2 = 90°$, and v is the number of feet per second in the initial velocity. Then

$$A = \int_0^{\frac{2v^2 \sin\theta \cos\theta}{g}} \frac{2v^2x \cos\theta \sin\theta - gx^2}{2v^2 \cos^2\theta}\, dx = \frac{2}{3}\frac{v^4 \sin^3\theta \cos\theta}{g^2},$$

giving the ratio

$$\frac{A_1}{A_2} = \frac{\sin^3\theta_1 \cos\theta_1}{\cos^3\theta_2 \sin\theta_2} = \tan^2\theta_1, \quad \text{since} \quad \theta_2 = \frac{\pi}{2} - \theta_1.$$

This also might have been gotten by dividing the maximum altitude of the first projectile, $(v^2 \sin^2\theta_1)/2g$, by that of the second projectile, $(v^2 \cos^2\theta_1)/2g$, as the horizontal distance of the two, $(2v^2 \cos\theta \sin\theta)/g$, is the same.

Also note that if $\theta_1 = \theta_2$, we have a maximum horizontal distance. And further, noting the equation, $y = vt \sin\theta - \frac{1}{2}gt^2$, we find the total time $t = (2v \sin\theta)/g$, and $t_1/t_2 = \tan\theta_1$.

Note. This problem was incorrectly listed under Geometry in the November, 1913, issue. It should have been under Mechanics. EDITORS.

Also solved by RICHARD MORRIS, J. L. RILEY, B. L. LIBBY, C. N. SCHMALL, A. M. HARDING, HORACE OLSON, S. W. REAVES, W. C. EELLS, and J. W. CLAWSON.

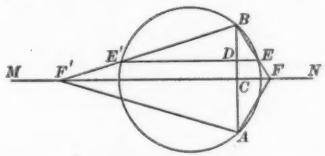
A solution of 424 by J. W. CLAWSON was received too late for publication in the March issue.

426. Proposed by R. D. CARMICHAEL, Indiana University.

On a given chord of a circle as a base construct an isosceles triangle, with vertex outside of the circle, such that its sides shall be divided in a given ratio by their points of intersection with the circle.

SOLUTION BY J. B. SMITH, Hampden-Sidney, Va.

Let AB be the given chord, C its mid-point and MN its perpendicular bisector. Let $m : n$ be the given ratio.



Divide the semi-chord BC in the given ratio. If D be the point of division, erect the perpendicular to AB at D. Let it cut the circle at E, E'; draw BE and produce it to meet MN at F and draw AF. Then AFB is the required triangle. For $AF = BF$ and $BE : EF = BD : DC = m : n$. $BF'A$ is another solution.

Also solved by G. W. HARTWELL, M. E. GRABER, C. HORNUNG, A. M. HARDING, ELMER SCHUYLER, RICHARD MORRIS, KENNETH REYNOLDS, BARNEM LIBBY, J. W. CLAWSON, and EMMA M. GIBSON.

CALCULUS.

341. Proposed by E. B. ESCOTT, University of Michigan.

Find the value for the volume of a barrel in terms of its length l, the bung diameter a and the head diameter b, also an approximate expression when a and b are nearly equal.

I. SOLUTION BY THE PROPOSER.

The simplest curve for the longitudinal cross section of the barrel is probably a parabola. Its equation, since it has its vertex on the y-axis and passes through the points $(-l/2, b/2)$, $(0, a/2)$, $(l/2, b/2)$, is

$$y = \frac{a}{2} - \frac{2(a-b)}{l^2}x^2.$$

The volume of solid obtained by revolving this curve around the x-axis is

$$\pi \int_{-\frac{l}{2}}^{\frac{l}{2}} y^2 dx = \pi \frac{l}{60} (8a^2 + 4ab + 3b^2).$$

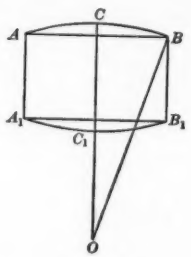
If a and b are nearly equal, $(a - b)^2$ is quite small. So the above expression for the volume may be replaced by

$$\pi \frac{l}{60} [8a^2 + 4ab + 3b^2 + 2(a - b)^2] = \pi \frac{l}{12} (2a^2 + b^2).$$

This latter formula can also be obtained from Simpson's Rule, $V = \frac{l}{6}(S_1 + 4S_2 + S_3)$ where S_1 and S_3 are end sections of the solid and S_2 the mid-section.

II. Solution by A. H. Holmes, Brunswick, Maine.

Let $ACBB_1C_1A_1$ be a longitudinal section of the barrel. Take O as the center of the circumference of a circle of which the arc ACB is a part. Put $OA = r$. The capacity of the barrel is equal to the sum of the contents of the cylinder ABB_1A_1 and the solid of revolution formed by the revolution of the segment ACB about the axis of the barrel. The distance of the center of gravity of the segment ACB from O is $\overline{AB}^3$ divided by 12 times the area of the segment. Therefore,



$$V = \frac{\pi l b^2}{4} + 2\pi \left\{ \frac{l^3}{12} - \left(r - \frac{a}{2}\right)\left[r^2 \sin^{-1} \frac{l}{2r} - \frac{l}{2}\left(r - \frac{a-b}{2}\right)\right]\right\}$$

where

$$r = \frac{l^2 + (a-b)^2}{4(a-b)}.$$

For a very close approximation when a and b are nearly equal, take ACB as an arc of an ellipse of which the bung diameter of the barrel is the minor axis. Then

$$V = 2\pi \int_0^{l/2} y^2 dx = 2\pi B^2 \int_0^{l/2} \left(1 - \frac{x^2}{A^2}\right) dx = 2\pi B^2 \left(\frac{l}{2} - \frac{l^3}{3A^2}\right),$$

where $B^2 = a^2/4$, and hence $A^2 = a^2 l^2/4(a^2 - b^2)$. Hence $V = \pi \frac{l}{12}(2a^2 + b^2)$.

MECHANICS.

276. Proposed by V. M. SPUNAR, Chicago, Illinois.

Find the center of gravity of the volume formed by the revolution around the x-axis of the area of the curve $y^4 - axy^2 + x^4 = 0$.

SOLUTION BY RICHARD MORRIS, Rutgers College.

From

$$y^2 = \frac{ax \pm x\sqrt{a^2 - 4x^2}}{2},$$

we find the x limits to be 0 and $a/2$.

Hence

$$\bar{x} = \frac{\int_0^{a/2} \pi(y_1^2 - y_2^2)x\,dx}{\int_0^{a/2} \pi(y_1^2 - y_2^2)\,dx}$$

or

$$\bar{x} = \frac{\int_0^{a/2} \left[\frac{ax + x\sqrt{a^2 - 4x^2} - ax + x\sqrt{a^2 - 4x^2}}{2}\right] x\,dx}{\int_0^{a/2} \left[\frac{ax + x\sqrt{a^2 - 4x^2} - ax + x\sqrt{a^2 - 4x^2}}{2}\right] dx} = \frac{\int_0^{a/2} x^2(a^2 - 4x^2)^{\frac{1}{2}}\,dx}{\int_0^{a/2} (a^2 - 4x^2)^{\frac{1}{2}} x\,dx}.$$

Putting $2x = z$, and transforming the numerator only, we get

$$\bar{x} = \frac{\int_0^{a} \frac{1}{8}z^2(a^2 - z^2)^{\frac{1}{2}}\,dz}{\int_0^{a/2} (a^2 - 4x^2)^{\frac{1}{2}} x\,dx},$$

which easily integrates by means of tables, giving

$$\bar{x} = \frac{\frac{1}{8}\left[\frac{z}{8}(2z^2 - a^2)\sqrt{a^2 - z^2} + \frac{a^4}{8}\sin^{-1}\frac{z}{a}\right]_0^a}{-\frac{1}{8}\left[(a^2 - 4x^2)^{\frac{3}{2}} \cdot \frac{2}{3}\right]_0^{a/2}} = \frac{3a\pi}{32}.$$

Also solved by S. W. REAVES and H. E. TREFETHEN.

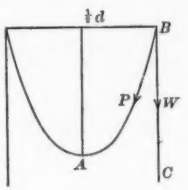
280. Proposed by C. N. SCHMALL, New York City.

Given the distance d between two smooth hooks in the same horizontal line. Show that the shortest string which can form a catenary, with these hooks for points of support, is de where e is the base of the Naperian system of logarithms.

SOLUTION BY A. M. HARDING, University of Arkansas.

Let $y = c/2(e^{\frac{x}{c}} + e^{-\frac{x}{c}})$ be the catenary. Let $BC = l$, then $W = wl$, where w = weight per unit length of the string. The tension at any given point of the catenary is given by $T \sin\theta = ws$, where s = the length of the string measured

from A and θ is the angle which the string makes with the horizontal at the given point. It can be easily shown that $\tan\theta = \sinh x/c$, so that $\sin\theta = \tanh x/c$, and hence $T = ws \coth x/c$.



At the point B we have $s = c \sinh d/2c$ and $T = P$, say. Since the hook is smooth $P = W$ and we have $wl = wc \sinh d/2c \coth d/2c = wc \cosh d/2c$. Hence $l = c \cosh d/2c$ and $\frac{1}{2}L = l + s = c(\cosh d/2c + \sinh d/2c) = ce^{\frac{d}{2c}}$.

We must now find the value of c which makes $\frac{1}{2}L$ a minimum. From the equation $D_c(\frac{1}{2}L) = 0$ we obtain

$$e^{\frac{d}{2c}} - \frac{d}{2c}e^{\frac{d}{2c}} = \tfrac{1}{2}e^{\frac{d}{2c}}\left(2 - \frac{d}{c}\right) = 0, \quad \text{or} \quad c = \frac{d}{2}.$$

Hence $\frac{1}{2}L = d/2 \cdot e = de/2$ or $L = de$.

Also solved by J. B. SMITH.

NUMBER THEORY.

187. Proposed by E. T. BELL, New York, N. Y.

If m is any integer, P the product of all the distinct prime factors of m and λ their number, and if $N(x)$ denote the number of divisors of x, then

$$6^{\lambda}\Sigma N(d)N(m/d) = N(m)N(Pm)N(P^2m),$$

where the sign of summation extends over all the divisors d of m.

SOLUTION BY THOMAS E. MASON, Indiana University.

Since m has λ distinct prime factors it can be written

$$m = p_1^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3}\cdots p_\lambda^{\alpha_\lambda},$$

where the p's are distinct primes. By a well-known formula we have

$$N(m) = (\alpha_1+1)(\alpha_2+1)\cdots(\alpha_\lambda+1), \quad N(Pm) = (\alpha_1+2)\cdots(\alpha_\lambda+2), \quad N(P^2m) = (\alpha_1+3)\cdots(\alpha_\lambda+3). \tag{1}$$

Any divisor d of m can be written in the form

$$d = p_1^{r_1}p_2^{r_2}p_3^{r_3}\cdots p_\lambda^{r_\lambda}.$$

Then

$$N(d) = (r_1+1)(r_2+1)\cdots(r_\lambda+1),$$

and

$$N(m/d) = (\alpha_1 - r_1 + 1)(\alpha_2 - r_2 + 1)\cdots(\alpha_\lambda - r_\lambda + 1).$$

Now we have

$$6^\lambda \Sigma N(d)N(m/d) = 6^\lambda \sum_{r_i=0}^{\alpha_i} (\alpha_1 - r_1 + 1)(\alpha_2 - r_2 + 1) \cdots (\alpha_\lambda - r_\lambda + 1)(r_1 + 1)(r_2 + 1) \cdots (r_\lambda + 1)$$

$$= 6^\lambda \sum_{r_i=0}^{\alpha_i} (\alpha_1 - r_1 + 1)(r_1 + 1) \cdots (\alpha_\lambda - r_\lambda + 1)(r_\lambda + 1) \tag{2}$$

$$= 6 \sum_{r_1=0}^{\alpha_1} [\alpha_1(r_1 + 1) - (r_1^2 - 1)] \cdot 6 \sum_{r_2=0}^{\alpha_2} [\alpha_2(r_2 + 1) - (r_2^2 - 1)] \cdots 6 \sum_{r_\lambda=0}^{\alpha_\lambda} [\alpha_\lambda(r_\lambda + 1) - (r_\lambda^2 - 1)].$$

The indicated summations can be readily performed. We have

$$\sum_{r_i=0}^{\alpha_i} (r_i + 1) = \frac{\alpha_i(\alpha_i + 1)}{2} + \alpha_i + 1 = \frac{(\alpha_i + 1)(\alpha_i + 2)}{2}$$

and

$$\sum_{r_i=0}^{\alpha_i} (r_i^2 - 1) = \frac{\alpha_i(\alpha_i + 1)(2\alpha_i + 1)}{6} - \alpha_i - 1 = \frac{(\alpha_i + 1)(\alpha_i + 2)(2\alpha_i - 3)}{6}.$$

Hence

$$6 \sum_{r_i=0}^{\alpha_i} [\alpha_i(r_i + 1) - (r_i^2 - 1)] = 6 \left[\alpha_i \frac{(\alpha_i + 1)(\alpha_i + 2)}{2} - \frac{(\alpha_i + 1)(\alpha_i + 2)(2\alpha_i - 3)}{6} \right]$$

$$= (\alpha_i + 1)(\alpha_i + 2)(\alpha_i + 3).$$

Substituting for these sums in the last member of (2) and making use of (1), we have

$$6^\lambda \Sigma N(d)N(m/d) = (\alpha_1 + 1)(\alpha_1 + 2)(\alpha_1 + 3)(\alpha_2 + 1)(\alpha_2 + 2)(\alpha_2 + 3) \cdots (\alpha_\lambda + 1)(\alpha_\lambda + 2)(\alpha_\lambda + 3)$$

$$= N(m)N(Pm)N(P^2m).$$

197. Proposed by E. T. BELL, Seattle, Washington.

Show that in the expansion of

$$\frac{1 + z + z^2 + \cdots + z^{p-1}}{(1 - z)^{p-1}} - 1,$$

where p is a prime, the coefficients of the various powers of z are divisible by p.

[Eisenstein, *Crelle*, t. 27, p. 282.]

SOLUTION BY B. F. YANNEY, University of Wooster.

The given expression is evidently equal to $(1 - z^p)/(1 - z)^p - 1$, which may be put in the form $(1 - z^p)(1 - z)^{-p} - 1$. Expanding the second factor of the first term and noticing that, since p is prime, the coefficient of each term

in the expansion except that of the 1st, the $(p+1)$th, the $(2p+1)$th, and so on to the $(np+1)$th, and so on, is divisible by p, we obtain, omitting the terms with the divisible coefficients,

$$(1-z^p)\left(1+\frac{p(p+1)\cdots(2p-1)}{p!}z^p+\frac{p(p+1)\cdots(3p-1)}{(2p)!}z^{2p}+\cdots\right.$$
$$\left.+\frac{p(p+1)\cdots((n+1)p-1)}{(np)!}\cdots\right)-1.$$

Performing the indicated operations, we get

$$\left(\frac{p(p+1)\cdots(2p-1)}{p!}-1\right)z^p+\left(\frac{p(p+1)\cdots(3p-1)}{(2p)!}-\frac{p(p+1)\cdots(2p-1)}{p!}\right)z^{2p}$$
$$+\cdots+\left(\frac{p(p+1)\cdots((n+1)p-1)}{(np)!}-\frac{p(p+1)\cdots(np-1)}{((n-1)p)!}\right)z^{np}+\cdots.$$

The first coefficient may be written $[(p+1)(p+2)\cdots(p+p-1)-(p-1)!]/(p-1)!$, which is equal to $[pA+(p-1)!-(p-1)!]/(p-1)!$, where A is a polynomial in p. This expression, which is equal to $pA/(p-1)!$, and is an integer, as are all the coefficients, is plainly divisible by p, since the denominator does not contain p as a factor.

The coefficient of the general term may be changed in form to

$$\frac{p(p+1)\cdots(np-1)}{((n-1)p)!}\left(\frac{np(np+1)\cdots(np+p-1)}{[(n-1)p+1][(n-1)p+2]\cdots[np]}-1\right).$$

The first factor of this is an integer; hence so is the second, which latter can be expressed in the form

$$\frac{(np+1)(np+2)\cdots(np+p-1)-[(n-1)p+1][(n-1)p+2]\cdots[(n-1)p+p-1]}{[(n-1)p+1][(n-1)p+2]\cdots[(n-1)p+p-1]},$$

which equals

$$\frac{(npB+(p-1)!)-((n-1)pC+(p-1)!)}{(n-1)pC+(p-1)!},$$

where B and C are polynomials in np and $(n-1)p$, respectively. Since the denominator does not contain p as a factor, but the numerator does, the expression is divisible by p, which completes the proof.

Also solved by ELMER SCHUYLER, H. C. FEEMSTER and the PROPOSER.

MISCELLANEOUS QUESTIONS.

EDITED BY R. D. CARMICHAEL.

Professor F. R. Moulton, of the University of Chicago, calls attention to the fact that the differential equation proposed by Mr. Louis Cohen (see our question 1 in the January issue) belongs to the class given general treatment by him and Dr. W. D. MacMillan in the American Journal of Mathematics, volume 33, pages 63 and 96.

For question 4, proposed in the issue of December, 1913, we need more comprehensive answers than any yet received. We have in hand an excellent reply to question 6; the topic suggested in it is of such interest, however, that several answers are desirable.

QUESTION.

10. What use has been made of regular conference periods for assistance to individual students of secondary and college mathematics, and what services may they render?

REPLY.

5. In what ways may mathematics contribute most to the culture of the individual? What is being done and what may be done to advantage in the matter of developing courses in culture mathematics?

I. REMARKS BY D. N. LEHMER, University of California.

There is, no doubt, a strong prejudice in the minds of many mathematicians against culture courses in mathematics. The phrase is often understood to apply to courses in which there is no attempt to insist on accurate thinking, and where the emphasis is laid not so much on mathematics as upon stories connected with mathematics. The student will learn what an extraordinary amount of time and attention was given by the Alexandrian Greeks to the problem of the trisection of an angle, but he will not know accurately what the problem really meant. As to the reason why their efforts were foredoomed to meet with failure, and just what was the difficulty with which they struggled, these matters will be passed over in haste and perhaps in silence. He will be familiar with all of the great names in mathematics, but he will not have a single clear idea of any of the problems which engaged their attention. These are the "culture courses" that have made the very name an abomination to thoughtful teachers.

The history of mathematics is a great and important subject, but any course based chiefly on that subject will almost certainly be a waste of time for the serious student. The result is not mathematical training at all, but an attempt at historical training and that in a domain where we have not yet developed the rigorous discipline to be found in some subjects. We are all of us familiar with the superficial talker who can converse fluently on the most abstruse subjects without having one clear idea in his head about them. Why should we go to the trouble to plan a course for the development and encouragement of just such creatures?

But there are students whose chief interests lie in other branches of science who would nevertheless like to have an adequate notion of the state of affairs in mathematics. The number of such students is not so small as some people may imagine, and there are enough of them who are ready to go to some trouble in applying themselves to the study to make it worth while to plan a course that shall not be microscopic, nor yet telescopic, and which shall put them in possession of the most important processes and developments of the subject. Such a course has been given at the University of California for a number of years, and while each class has been made the subject of careful experiment, the courses as a whole have been pretty much the same from year to year.

The importance of the notion of one-to-one correspondence in all branches of knowledge would seem to demand that the relation between algebra and geometry be made clear to all serious students. Accordingly it is usual to start the years' work with a drill in the fundamental notions in analytic geometry. The students are freshmen and some may be lacking in trigonometry, but it is possible to teach analytic geometry with a very little trigonometry, and such as is absolutely essential can be supplied in a week or two as is necessary. The circle and conic sections are touched on very lightly, if at all, and progress is made towards the notion of the calculus by means of curves defined by equations of the form $y =$ polynomial in x. Such curves are especially useful in introducing the idea of the derivative, maxima and minima, inflection points, integration and evaluation of areas, volumes, etc. It is believed that it is of considerable importance that the student get the fundamental notions of the calculus with a minimum of effort in the way of actual manipulation of analytic expressions. Even the formal notation used in books on the calculus would seem to be of secondary importance. A student may emerge from such a course never having heard of a reduction formula, but he will know how to state a problem in finding areas or volumes in such a way that it will mean for him the discovery of an anti-derivative.

The course lasts for the whole year, three hours per week. It is possible with some classes to develop certain other ideas such as the foundations of number, or the processes of synthetic projective geometry. The chief emphasis is laid, however, on the principles of the calculus. It is in no sense a "snap" course, and its success is abundantly vouched for in the number of those who come to taste and remain to pursue the subject through the rest of their college career.

This is our notion of a "culture course," and while it is sometimes thought of by the ill-informed as a "hodge-podge" because we do not separate trigonometry and analytic geometry and the calculus into water-tight compartments, it serves the purpose of informing the student of the most important processes of modern mathematics, and at the same time demands from him his utmost efforts.

NOTES AND NEWS.

FLORIAN CAJORI, CHAIRMAN OF COMMITTEE.

Dr. OSCAR PERRON, who since 1910 has been professor of mathematics at Tübingen, is called to a professorship at Heidelberg.

Professor ARNOLD EMCH, of the University of Illinois, has accepted a position on the summer session faculty of the University of California for the coming summer.

Western Reserve University announces the continuation of the teachers' class in mathematics into the second semester. This takes the form of a course in the history of mathematics, conducted by Mr. ALVA H. FORD, of the department of mathematics of Adelbert College.

Professor ANTONIO FAVARO, the well-known Italian historian of mathematics and science, has recently made a biographical study of Niccolò Tartaglia. The results of his researches are published in *Isis* for November, 1913, and in a pamphlet, *Per la biografia di Niccolò Tartaglia*, Firenze, 1913.

The Macmillan Company announces for spring publication "The Geometry of Four Dimensions," by HENRY PARKER MANNING, and "Memorabilia Mathematica. The Philomath's Quotation Boook," by ROBERT E. MORITZ, of the University of Washington.

The paper on "Science and mathematics in vocational schools," which was read by Professor FLORIAN CAJORI, of Colorado College, before the Central Association of Science and Mathematics Teachers at Des Moines, Iowa, on November 28, has been published in the February number of *School Science and Mathematics*.

The "History of Japanese Mathematics," by D. E. SMITH and YOSHIO MIKAMI, announcement of which was made in the MONTHLY some time since, has appeared from the press of the Open Court Publishing Co. This work places within reach of English readers a field of mathematical history which hitherto has been quite inaccessible.

The thirty-fourth regular meeting of the Chicago Section of the American Mathematical Society was held at the University of Chicago on Friday and Saturday, April 10, 11, 1914. There were 74 persons in attendance upon the various sessions, including 47 members of the Society. This was the second meeting of the Section which was designated as a regular meeting of the Society at Chicago. The first was in December, 1913.

The February number of the *School Review* contains an article on "A plan for testing methods of teaching elementary mathematics," by Professor G. W. MYERS, of the School of Education, University of Chicago. It contains an account of an interesting experiment made by a high school teacher who undertook to test systematically the heuristic method as against the expository method of teaching first year algebra. He chose for the experiment two of his first year

algebra classes that were, in his judgment, similarly circumstanced and nearly equal in ability. One class was taught by the heuristic method as nearly as the teacher could administer it, and the other was taught entirely by the plan of direct exposition. Mr. Myers makes a plea for more experiments of this kind.

On January 12 occurred the death of CALVIN MILTON WOODWARD, professor emeritus of mathematics and dean of the school of engineering at Washington University, St. Louis, Mo. He was the originator and director of the St. Louis Manual Training School. One of his last important publications is a text on "Applied Mechanics for Engineering and Architectural Students." Professor Woodward was an active and influential member of various scientific and educational organizations, and was once president of the American Association for the Advancement of Science.

Two fellowships in the department of mathematics and two in the department of physics at the Rice Institute will be filled for the academic year 1914–15, each of the annual value of seven hundred and fifty dollars ($750). The successful candidates will be expected to enter upon a course of study and research work leading to the degree of Doctor of Philosophy, and also to assist with elementary teaching in mathematics or physics for about six hours per week. Applications accompanied by testimonials and a full statement of previous work and training should be addressed to the department of mathematics or to the department of physics, The Rice Institute, Houston, Texas.

Mr. CLIFFORD N. MILLS secured the degree of A.M. at the University of Indiana at the close of the last semester and is now acting professor of mathematics at the South Dakota State College at Brookings, in the place of Dr. G. L. Brown, who has been made acting president.

HENRY FREDERICK BAKER, Sc.D., F.R.S., has been elected Lowndean professor of astronomy and geometry at the University of Cambridge in succession to the late Sir Robert Ball. Before this election Dr. Baker was fellow and lecturer of St. John's College, and Cayley university lecturer in mathematics at Cambridge. He has held the presidency of the London Mathematical Society and of the Cambridge Philosophical Society.

In recent years several German professors of mathematics have called public attention to the fact that the number of mathematical students at the various German universities is larger than the probable number of mathematical positions in the German schools. According to the *Jahresbericht der Deutschen Mathematiker-Vereinigung*, volume 22 (1913), page 369, the number of mathematical students is much smaller during the current year than it has been during recent years. The number of women students of mathematics is, however, still on the increase in the German universities.

July 24, 1914, is the date for the Napier Tercentenary Celebration, to be held in Edinburgh, Scotland. The celebration will be held under the auspices of the Royal Society of Edinburgh. We quote from a recent circular the following: " The Celebration will be opened on Friday with an Inaugural Address by Lord

of Appeal Sir J. Fletcher Moulton, F.R.S., LL.D., followed by a reception given by the Right Honourable the Lord Provost, Magistrates, and Council of the City of Edinburgh. On Saturday and Monday the historical and present practice of computation and other developments closely connected with Napier's discoveries and inventions will be discussed. . . . Among many who have expressed a warm interest in the Celebration and who hope to take part in the Congress, may be mentioned Professor Andoyer, Paris; Professor J. Bauschinger, Strassburg; Professor Hume Brown, Historiographer Royal for Scotland; Professor Florian Cajori, Colorado, U. S. A.; Professor G. A. Gibson, Glasgow; Dr. J. W. L. Glaisher, Cambridge; Professor Lang, St. Andrews; Professor Macdonald, Aberdeen; Professor E. Pascal, Naples; Professor Karl Pearson, London; Professor David Eugene Smith, New York; Professor Steggall, Dundee; Professor Whittaker, Edinburgh. . . . Relics of Napier, collected by Lord Napier and Ettrick and other representatives of the family, will also be on view; and it is intended to bring together for exhibition books of Tables and forms of Calculating Machines, which may reasonably be regarded as natural developments of the great advance made by Napier." It is planned to issue a memorial volume containing the addresses and papers read before the Congress, and other material of historic and scientific value. Mathematicians from all countries are invited to attend the celebration.

It will be recalled that the floods last year prevented the Ohio teachers of mathematics and science from holding their annual meeting. Hence the following report from Professor S. E. Rasor, Vice-President of the association, is most gratifying.

"The tenth annual meeting of the Association of Ohio Teachers of Mathematics and Science was held at the Ohio State University, Columbus, Ohio, Friday evening, April 3 and Saturday, April 4. The Friday evening address was given by James F. Barker of the Cleveland East Technical High School. On Saturday the mathematics section listened with much interest and lasting impression to the addresses by Professor H. E. Slaught of the University of Chicago. Papers and discussions were also presented before this section by a number of prominent Ohio teachers. A large and most enthusiastic and representative body of teachers from the colleges and secondary schools of the state were present and declared the meeting to be the best in the history of the Association. A significant feature of the meeting was the fact that of those present about ninety-five per cent. were men. The members of the association were guests of the University at a complimentary noon-day luncheon on Saturday. Officers for the coming year were elected as follows: President, C. C. Morris, Ohio State University, Columbus, Ohio; Vice-President, S. J. Mauckley, Woodward High School, Cincinnati, Ohio; Secretary-Treasurer, E. W. E. Schear, Ottebein University, Westerville, Ohio; Assistant Secretary, Miss Margaret Devereaux, Alliance High School, Alliance, Ohio."

The twenty-sixth educational conference of the academies and high schools in relation with the University of Chicago was held at the University on Friday

and Saturday, April 17–18, 1914. At the mathematics section two leading questions were discussed (1) The comparative merits of home study and supervised study at the school; (2) Reviews of mathematical literature for teachers.

The first topic was presented by Mr. C. M. Austin, of the Oak Park High School, who gave a resumé of an extended paper on this subject by Mr. E. R. Breslich, based upon a careful investigation at the high school connected with the University of Chicago School of Education.

The second topic was introduced by Mr. H. C. Wright, of the University High School and Professor H. E. Slaught, of the University of Chicago. Mr. Wright presented a resumé of the chief articles published in the *Mathematics Teacher* during the past six years, and Professor Slaught explained the character of the literature published in the MONTHLY since it was reorganized in January, 1913, and dedicated to the interests of teachers of mathematics in the early collegiate and advanced secondary fields. The fact was brought out and emphasized that in our associations of secondary teachers of mathematics the discussions during the past ten or twelve years have dealt primarily with pedagogical and administrative schemes and that the journals representing these associations, such as the *Mathematics Teacher* and *School Science and Mathematics*, have properly and naturally reflected these aspects. On the other hand, little or no attention has been given to the *teachers themselves* who are by far the most important factor in any educational scheme, and who need sources of inspiration and power outside of their daily routine. Such inspiration and quickening of enthusiasm the AMERICAN MATHEMATICAL MONTHLY is trying to provide and it is the only journal in this country occupying this field. Special reference was made to the study of synthetic projective geometry as absolutely essential to provide a background and a viewpoint for the teacher of elementary plane geometry. It is believed that a text-book will be soon forthcoming which will be suitable not only for use in first or second year college classes, by those who are preparing to teach, but also for private study in small groups by teachers now in the service.

Errata. The following errors have been noted in the December issue, 1913. On page 304, line 10 up should read $z^3 = -6r\rho^2$; line 8 up should read $\rho = \pm 6k\sigma^3$; line 5 up should read $y = p^3 \mp 6\sigma^3$. On page 305, line 19 up, the first double sign should be $\mp$.

All copies of the MONTHLY for January, 1913, have been exhausted. The demand for sample copies was so great for this particular number that the supply was entirely inadequate. Any one who may know of extra copies not belonging to sets will confer a great favor by informing the MANAGING EDITOR.

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ON SOME GEOMETRIC PROPERTIES OF CIRCULAR TRANSFORMATIONS.

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§ 1. **Introduction.** In what follows I shall give simple demonstrations of a few geometric theorems that are of value in the study of ring-shaped domains as used in the definition of certain special automorphic functions.[1] As will be seen later on, the theory of such domains is closely connected with linear substitutions or circular transformations in a complex plane, as defined in § 3. Although there is nothing essentially new as to content, it will be at once apparent, how simply, in comparison with ordinary analytic methods, some of the theorems may be proved by making use of a fundamental proposition in group-theory. With this fact I want to emphasize the importance of an early introduction of linear substitutions and their principal group-properties in certain mathematical courses, as for example in function-theory.

A *ring-shaped domain* G may be defined as a connected portion of a complex plane bounded by two non-intersecting circles (see Fig. p. 140). Among such domains are included those into which G passes when the two circles become tangent.

By a *group* we understand a class of operations, such that the product of any two operations of the class is again an operation of the same class. The interpretation of this definition for linear substitutions will be found in § 3.

§ 2. **Reflexion on a Circle.** By *reflexion* of a point P on *a straight line* l we define a point P' on a line through P perpendicular to l, such that the distances of the distinct points P and P' from l are the same. The transformation by reciprocal radii with respect to a fixed circle is called *reflexion on the circle.*

An *inversion* with respect to a fixed circle may be defined as a reflexion on that circle followed by a reflexion on a fixed axis through the center of the same circle. This circle may, of course, be located anywhere in the plane.

[1] SCHOTTKY: "Ueber die Funktionenklasse die der Gleichung $F(\alpha x + \beta/\gamma x + \delta) = F(x)$ genügt," *Crelle's Journal*, Vol. 143, pp. 1–24 (June, 1913). See also the same *Journal*, Vol. 101, pp. 231–236 (1887).